

REPRESENTATION OF MEAN-PERIODIC FUNCTIONS IN SERIES OF EXPONENTIAL POLYNOMIALS

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ABSTRACT. Let θ be a Young function and consider the space $\mathcal{F}_\theta(\mathbb{C})$ of all entire functions with θ -exponential growth. In this paper, we are interested in the solutions $f \in \mathcal{F}_\theta(\mathbb{C})$ of the convolution equation $T \star f = 0$, called mean-periodic functions, where T is in the topological dual of $\mathcal{F}_\theta(\mathbb{C})$. We show that each mean-periodic function can be represented in an explicit way as a convergent series of exponential polynomials.

1. INTRODUCTION

A periodic function f with period t may be defined in terms of convolution equation as a function verifying

$$(\delta_t - \delta_0) \star f = 0,$$

while a function with zero average over an interval of length $t > 0$ satisfies the convolution equation

$$\mu \star f = 0,$$

where μ is defined by $\langle \mu, f \rangle = \frac{1}{t} \int_{-t/2}^{t/2} f(x) dx$. From the observation that the second notion is more natural from the point of view of experimental physics, Delsartes generalized the concept of periodic functions by introducing in [7] the notion of "mean-periodic" functions as the solutions of homogeneous convolution equations.

In this paper, we are dealing with the problem of representing mean periodic functions as series of exponentials polynomials.

Let us denote by $\mathcal{H}(\mathbb{C})$ the space of all entire functions on \mathbb{C} . Let θ be a Young function and θ^* its Legendre transform (see Definitions 2.1 and 2.2 below). The mean-periodic functions will lie in the space $\mathcal{F}_\theta(\mathbb{C})$ of all functions $f \in \mathcal{H}(\mathbb{C})$ such that

$$(1) \quad \sup_{z \in \mathbb{C}} |f(z)| e^{-\theta^*(m|z|)} < \infty,$$

for all constants $m > 0$.

We will also consider the limit case where $\theta(x) = x$. In this case, the associated conjugate function θ^* is formally infinite. Therefore, no growth condition of the type (1) is involved and we put $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{H}(\mathbb{C})$.

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We will say that $f \in \mathcal{F}_\theta(\mathbb{C})$ is a mean-periodic function if, for a certain non zero analytic functional $T \in \mathcal{F}'_\theta(\mathbb{C})$, f verifies the convolution equation

$$(2) \quad T \star f = 0.$$

We will then say that f is T -mean-periodic.

For example, if we denote by $\{\alpha_k\}_k$ the zeros of the Fourier-Borel transform of T and m_k their order of multiplicity, then all exponential monomials $z^j e^{\alpha_k z}$, with $j < m_k$ are T -mean-periodic functions (see Lemma 3.4). Then, every convergent series whose general term is a linear combination of such exponential monomials is also a T -mean-periodic function.

Our main result (see Theorem 3.5) states roughly that the converse holds, provided that we apply an Abel summation procedure in order to make the sum convergent. In fact, we prove that any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(3) \quad f(z) = \sum_k \sum_{l=0}^{m_k-1} c_{k,l} \left[\sum_{j=0}^k e^{z\alpha_j} P_{k,j,l}(z) \right],$$

where $P_{k,j,l}$ are polynomials of degree $< m_j$, explicitly given by (16) and (17) in terms of $\{(\alpha_k, m_k)\}_k$. Moreover, the coefficients $c_{k,l}$ verify the growth condition (8) and can be explicitly computed in terms of f and T .

When $V = \{(\alpha_k, m_k)\}_k$ is an interpolating variety (see Definition 5.1), no Abel summation process is needed, we simply obtain that any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(4) \quad f(z) = \sum_k e^{z\alpha_k} \sum_{l=0}^{m_k-1} d_{k,l} \frac{z^l}{l!},$$

where the coefficients $d_{k,l}$ verify the growth estimate (19) (see Theorem 5.2).

Our present work is inspired by the paper [6], written by C.A. Berenstein and B.A. Taylor in 1975 where the authors considered the case $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{H}(\mathbb{C})$. In fact, they showed that, given $T \in \mathcal{H}'(\mathbb{C})$, there exists a sequence of indices $k_0 = 0 < k_1 < \dots < k_n < \dots$ such that any T -mean periodic function $f \in \mathcal{H}(\mathbb{C})$ admits a unique expansion, convergent in $\mathcal{H}(\mathbb{C})$, of the form

$$(5) \quad f(z) = \sum_n \sum_{k_n \leq k < k_{n+1}} e^{z\alpha_k} \sum_{j=0}^{m_k-1} d_{k,j} \frac{z^j}{j!}.$$

In (5), the sum converges by packets grouping instead of Abel summation. But in general, the sequence $\{k_n\}_n$ is not explicit, except in the case when V is an interpolating variety, where the sequence $k_n = n$ works, thus formula (5) leads to (4).

In 1988, representation formulas of the form (5) were given by C.A. Berenstein and D.C. Struppa (cf [4]) in the case where $\theta(x) = x^p$, $p > 1$ and in the more complicated situation where $\mathcal{F}_\theta(\mathbb{C})$ is replaced by $\mathcal{F}_\theta(\Gamma)$ with Γ an open convex cone in \mathbb{C} , provided some natural conditions on the behavior of the Fourier-Borel transform of T .

We also refer the interested reader to [5] for a general survey on the connections between mean periodicity and complex analysis in \mathbb{C}^n .

To conclude the introduction, here is how the paper is organized : Section 2 is devoted to preliminary definitions and useful results from functional analysis. The main results are stated in section 3 and their proofs are given in section 4. Finally, in section 5, we study the particular case when V is an interpolating variety.

2. PRELIMINARIES AND DEFINITIONS.

Definition 2.1. A function $\theta : [0, +\infty[\rightarrow [0, +\infty[$ is called a Young function if it is convex, continuous, increasing and verifies $\theta(0) = 0$ and $r = o(\theta(r))$ when $r \rightarrow +\infty$.

Definition 2.2. Let θ be a Young function. The Legendre transform θ^* of θ is the function defined by

$$\theta^*(x) = \sup_{t \geq 0} (tx - \theta(t)).$$

Note that the Legendre transform of a Young function is itself a Young function and $\theta^{**} = \theta$. We refer the reader to [10] for further details. Throughout the paper, θ will denote either the function $\theta(x) = x$ or a Young function.

For any $m > 0$, consider $E_{\theta,m}(\mathbb{C})$, the Banach space of all functions $f \in \mathcal{H}(\mathbb{C})$ such that

$$\|f\|_{\theta,m} := \sup_{z \in \mathbb{C}} |f(z)| e^{-\theta(m|z|)} < +\infty$$

and define

$$\mathcal{G}_\theta(\mathbb{C}) = \cup_{p \in \mathbb{N}^*} E_{\theta,p}(\mathbb{C})$$

endowed with the inductive limit topology. It is clear that $\mathcal{G}_\theta(\mathbb{C})$ is an algebra under the ordinary multiplication of functions.

Remark 2.3. When $\theta(x) = x^k$, $k \geq 1$, $\mathcal{G}_\theta(\mathbb{C})$ is the space of all entire functions, either of order $< k$ or of order k and finite type. In particular, when $k = 1$, $\mathcal{G}_\theta(\mathbb{C})$ is the space of all entire functions of exponential type, usually denoted by $\text{Exp}(\mathbb{C})$.

We define the space $\mathcal{F}_\theta(\mathbb{C})$ as follows :

(i) In the case where $\theta(x) = x$, we put $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{H}(\mathbb{C})$, the space of all entire functions endowed with the topology of uniform convergence on every compact of \mathbb{C} . It is a Fréchet-Schwartz space (see [2]).

(ii) In the case where θ is a Young function, we denote

$$\mathcal{F}_\theta(\mathbb{C}) = \cap_{p \in \mathbb{N}^*} E_{\theta^*,1/p}(\mathbb{C})$$

endowed with the projective limit topology. The space $\mathcal{F}_\theta(\mathbb{C})$ is a nuclear Fréchet space (see [8]), hence it is a Fréchet-Schwartz space (see [13]).

For any fixed $\xi \in \mathbb{C}$, and $l \in \mathbb{N}$, we will denote by $M_{l,\xi}$ the exponential monomial $z \rightarrow z^l e^{\xi z}$. It is easy to see that $M_{l,\xi} \in \mathcal{F}_\theta(\mathbb{C})$. In the next we denote by $\mathcal{F}'_\theta(\mathbb{C})$ the strong topological dual of $\mathcal{F}_\theta(\mathbb{C})$.

Let us recall some definitions and properties from functional analysis. We refer to [2] for further details in the case (i) and to [8] for the case (ii).

To any fixed $u \in \mathbb{C}$, define the translation operator τ_u on $\mathcal{F}_\theta(\mathbb{C})$ by

$$(\tau_u f)(z) = f(z + u), \text{ for all } f \in \mathcal{F}_\theta(\mathbb{C}) \text{ and } z \in \mathbb{C}.$$

It's easy to see that $\mathcal{F}_\theta(\mathbb{C})$ is invariant under these translation operators.

For all $S \in \mathcal{F}'_\theta(\mathbb{C})$ and $f \in \mathcal{F}_\theta(\mathbb{C})$, the function $z \rightarrow \langle S, \tau_z f \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the duality bracket, is an element of $\mathcal{F}_\theta(\mathbb{C})$. Therefore, for any $S \in \mathcal{F}'_\theta(\mathbb{C})$, the map $S \star : \mathcal{F}_\theta(\mathbb{C}) \rightarrow \mathcal{F}_\theta(\mathbb{C})$ defined by

$$S \star f(z) = \langle S, \tau_z f \rangle$$

is a convolution operator, i.e., it is linear, continuous and commute with any translation operator.

For any $S \in \mathcal{F}'_\theta(\mathbb{C})$, the Fourier-Borel transform of S , denoted by $\mathcal{L}(S)$ is defined by

$$\mathcal{L}(S)(\xi) = \langle S, e^{\xi \cdot} \rangle,$$

where $e^{\xi \cdot} = M_{0,\xi}$ is the function $z \in \mathbb{C} \rightarrow e^{\xi z}$.

For any two elements S and U of $\mathcal{F}'_\theta(\mathbb{C})$, the convolution product $S \star U \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\forall f \in \mathcal{F}_\theta(\mathbb{C}), \quad \langle S \star U, f \rangle = \langle S, U \star f \rangle.$$

Moreover for any $S, U \in \mathcal{F}'_\theta(\mathbb{C})$

$$\mathcal{L}(S \star U) = \mathcal{L}(S)\mathcal{L}(U)$$

Under this convolution, $\mathcal{F}'_\theta(\mathbb{C})$ is a commutative algebra admitting δ_0 , the Dirac measure at the origin, as unit.

Proposition 2.4. *The Fourier-Borel transform \mathcal{L} is a topological isomorphism between the algebras $\mathcal{F}'_\theta(\mathbb{C})$ and $\mathcal{G}_\theta(\mathbb{C})$.*

3. MAIN RESULTS

Throughout the rest of the paper, let T be a fixed non-zero element of $\mathcal{F}'_\theta(\mathbb{C})$. Our main goal in this section is to show that any function $f \in \mathcal{F}_\theta(\mathbb{C})$ satisfying the equation

$$(6) \quad T \star f = 0$$

can be represented as convergent series of exponential-polynomials which are them-selves solution of (6).

Definition 3.1. We say that a function $f \in \mathcal{F}_\theta(\mathbb{C})$ is " T -mean-periodic" if it satisfies the equation (6).

Denote by Φ the entire function in $\mathcal{G}_\theta(\mathbb{C})$ defined by $\Phi = \mathcal{L}(T)$. Before going further, let us show the following division property :

Lemma 3.2. *Let $h \in \mathcal{H}(\mathbb{C})$ and $g \in \mathcal{G}_\theta(\mathbb{C})$. If g is not identically zero and if $f = gh \in \mathcal{G}_\theta(\mathbb{C})$, then $h \in \mathcal{G}_\theta(\mathbb{C})$.*

Proof. Up to a translation, we may assume that $g(0) \neq 0$. Let us apply the minimum modulus theorem and its corollary given in [2, Lemma 2.2.11] to the function g in the disc of center 0 and radius $2^{n+1}e$, where n is any positive integer.

As $g \in \mathcal{G}_\theta(\mathbb{C})$, there exists $p \in \mathbb{N}^*$ and $C_p > 0$ (not depending on n) such that

$$\max_{|\xi| \leq 2^{n+3}e} |g(\xi)| \leq C_p e^{\theta(p2^n)}.$$

Thus, there exists $\varepsilon_p > 0$ (not depending on n) and $R_n, 2^n \leq R_n \leq 2^{n+1}$ such that

$$\min_{|\xi|=R_n} |g(\xi)| \geq \varepsilon_p e^{-\theta(p2^n)}.$$

Let $n \in \mathbb{N}$ and $|\xi| = R_n$. As $f \in \mathcal{G}_\theta(\mathbb{C})$, there exists $q > 0$ and $C_q > 0$ (not depending on n), such that

$$|f(\xi)| \leq C_q e^{\theta(q2^n)}.$$

Using the convexity of θ and the fact that $\theta(0) = 0$, we have

$$\theta(p2^n) \leq \frac{1}{2}\theta(p2^{n+1}).$$

If we assume, for example, that $p \geq q$, we deduce that

$$|h(\xi)| = |f(\xi)| \frac{1}{|g(\xi)|} \leq \frac{C_q}{\varepsilon_p} e^{\theta(q2^n) + \theta(p2^n)} \leq B_p e^{\theta(p2^{n+1})}.$$

Now let $z \in \mathbb{C}$, be such that $2^{n-1} \leq |z| \leq 2^n \leq R_n$. By the maximum modulus theorem,

$$|h(z)| \leq B_p e^{\theta(p2^{n+1})} \leq B_p e^{\theta(4p|z|)}.$$

This proves that $h \in \mathcal{G}_\theta(\mathbb{C})$. ■

Corollary 3.3. *In the case where Φ has no zeros, the only mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ is the zero function.*

Proof. Assume that Φ has no zeros. Then $\frac{1}{\Phi} \in \mathcal{H}(\mathbb{C})$ and, by Lemma 3.2, $\frac{1}{\Phi} \in \mathcal{G}_\theta(\mathbb{C})$. By Proposition 2.4, $S = (\mathcal{L})^{-1}(\frac{1}{\Phi}) \in \mathcal{F}'_\theta(\mathbb{C})$. Then, we have $S \star T = T \star S = \delta_0$. If we assume $T \star f = 0$, then $\delta_0 \star f = f = 0$. ■

We will throughout the rest of the paper assume that Φ has zeros, and denote them by $|\alpha_0| \leq |\alpha_1| \leq \dots \leq |\alpha_k| \leq \dots$, $\alpha_k \neq \alpha_{k'}$ if $k \neq k'$.

We will denote by m_k be the order of multiplicity of Φ at α_k and we will consider the "multiplicity variety" $V = \{(\alpha_k, m_k)\}_{k \in \mathbb{N}}$ (see [1] for an introduction to the concept of multiplicity variety). We may use the notation $V = \Psi^{-1}(0)$.

Lemma 3.4. (i) *For all $\xi \in \mathbb{C}$ and $l \in \mathbb{N}$, we have $\langle T, M_{l,\xi} \rangle = \Phi^{(l)}(\xi)$.*

(ii) *Each exponential monomial M_{l,α_k} , for $0 \leq l < m_k$ is T -mean-periodic.*

Proof. To prove (i), we proceed by induction on $l \geq 0$. The property is true for $l = 0$ by definition of the Fourier-Borel transform of T .

Suppose the property true for l . Let $\xi \in \mathbb{C}$ be fixed. Let us verify that the function $\frac{M_{l,\xi+u} - M_{l,\xi}}{u}$ converges in $\mathcal{F}_\theta(\mathbb{C})$ to $M_{l+1,\xi}$ when u tends to 0. For all $z \in \mathbb{C}$ and $u \leq 1$, we have

$$\left| \frac{e^{uz} - 1}{u} - z \right| = \left| uz^2 \sum_{n \geq 2} \frac{(uz)^{n-2}}{n!} \right| \leq |u||z|^2 e^{|z|}.$$

This implies that

$$\left| \frac{M_{l,\xi+u}(z) - M_{l,\xi}(z)}{u} - M_{l+1,\xi}(z) \right| = |z|^l e^{\xi z} \left| \frac{e^{uz} - 1}{u} - z \right| \leq |u||z|^{l+2} e^{(1+|\xi|)|z|}.$$

Therefore, $\frac{M_{l,\xi+u} - M_{l,\xi}}{u}$ converges to $M_{l+1,\xi}$ for the topology of $\mathcal{F}_\theta(\mathbb{C})$ when u tends to 0.

From this, we obtain

$$\begin{aligned} \Phi^{(l+1)}(\xi) &= \lim_{u \rightarrow 0} \frac{\Phi^{(l)}(\xi + u) - \Phi^{(l)}(\xi)}{u} = \lim_{u \rightarrow 0} \frac{\langle T, M_{l,\xi+u} \rangle - \langle T, M_{l,\xi} \rangle}{u} \\ &= \lim_{u \rightarrow 0} \langle T, \frac{M_{l,\xi+u} - M_{l,\xi}}{u} \rangle = \langle T, M_{l+1,\xi} \rangle, \end{aligned}$$

by continuity of T . This completes the proof of (i).

In order to prove (ii), it is sufficient to see that

$$T \star M_{l,\alpha_k}(z) = \langle T, \tau_z M_{l,\alpha_k} \rangle = e^{\alpha_k z} \sum_{n=0}^l C_l^n z^{l-n} \langle T, M_{n,\alpha_k} \rangle = e^{\alpha_k z} \sum_{n=0}^l C_l^n z^{l-n} \Phi^{(n)}(\alpha_k).$$

■

Our main theorem states roughly that T -mean-periodic functions are series of linear combinations of the exponential monomials M_{l,α_k} :

Theorem 3.5. (i) Any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(7) \quad f(z) = \sum_{k \geq 0} \sum_{l=0}^{m_k-1} c_{k,l} \left[\sum_{j=0}^k e^{z\alpha_j} P_{k,j,l}(z) \right],$$

where $P_{k,j,l}$ are the polynomials of degree $< m_j$ given by (16) and (17). The coefficients $c_{k,l}$ verify the following estimate

$$(8) \quad \forall m > 0, \quad \sum_{k \geq 0} e^{\theta(m|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)} \right) < +\infty$$

and are given by

$$c_{k,l} = \langle S_{k,l}, f \rangle$$

where $S_{k,l} \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(S_{k,l})(\xi) = (\xi - \alpha_k)^l \prod_{n=0}^{k-1} (\xi - \alpha_n)^{m_n}.$$

(ii) Conversely, any such serie whose coefficients $c_{n,l}$ satisfy the estimate (8) converges in $\mathcal{F}_\theta(\mathbb{C})$ to a function f solving the equation (6).

Corollary 3.6. Assume that all the multiplicities m_k are equal to 1. Then

(i) any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(9) \quad f(z) = \sum_{k \geq 0} c_k \left[\sum_{j=0}^k e^{z\alpha_j} \prod_{0 \leq n \leq k, n \neq j} (\alpha_j - \alpha_n)^{-1} \right],$$

where the coefficients c_k satisfy the following estimate

$$(10) \quad \forall m > 0, \quad \sum_{k \geq 0} e^{\theta(m|\alpha_k|)} |c_k| (|\alpha_k| + 1)^{-k} < +\infty$$

and are given by

$$c_k = \langle S_k, f \rangle$$

where $S_k \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(S_k)(\xi) = \prod_{n=0}^{k-1} (\xi - \alpha_n)^{m_n}.$$

(ii) Conversely, any such series whose coefficients c_k satisfy the estimate (10) converges in $\mathcal{F}_\theta(\mathbb{C})$ to a function f solving the equation (6).

4. PROOF OF THEOREM 3.5

We define the restriction operator ρ on $\mathcal{G}_\theta(\mathbb{C})$ by

$$\rho(g) = \left\{ \frac{g^l(\alpha_k)}{l!} \right\}_{k, 0 \leq l < m_k}, \quad g \in \mathcal{G}_\theta(\mathbb{C}).$$

As an immediate consequence of Lemma 3.2, we have the following lemma.

Lemma 4.1. The kernel of the restriction operator ρ is the ideal generated by Φ in $\mathcal{G}_\theta(\mathbb{C})$, i.e.,

$$\text{Ker } \rho = \{\Phi g, g \in \mathcal{G}_\theta(\mathbb{C})\}.$$

We are going to use a characterization, obtained in [12], of the elements $a = \{a_{k,l}\}_{k, 0 \leq l < m_{k-1}}$ belonging to $\rho(\mathcal{G}_\theta(\mathbb{C}))$. This characterization is given in terms of growth conditions involving the divided differences (see [9] for further details about divided differences).

To any discrete doubly indexed sequence $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ of complex numbers, we associate the sequence of divided differences $\Psi(a) = \{b_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$. We recall that they are the coefficients of the Newton polynomials,

$$(11) \quad Q_q(\xi) = \sum_{k=0}^q \prod_{n=0}^{k-1} (\xi - \alpha_n)^{m_n} \left(\sum_{l=0}^{m_k-1} b_{k,l} (\xi - \alpha_k)^l \right),$$

defined, for any $q \in \mathbb{N}$, as the unique polynomial of degree $m_0 + \dots + m_q - 1$ such that

$$\frac{Q_q^{(l)}(\alpha_k)}{l!} = a_{k,l}, \quad \text{for } 0 \leq k \leq q \text{ and } 0 \leq l \leq m_k - 1.$$

When all the multiplicities $m_k = 1$, we may give a simple formula for the coefficients b_k :

$$b_k = \sum_{j=0}^k a_j \prod_{0 \leq n \leq k, n \neq j} (\alpha_j - \alpha_n)^{-1}.$$

In the general case (see [12]) we may define them by induction by :

$$b_{0,l} = a_{0,l}, \text{ for all } 0 \leq l < m_0,$$

and for $k \geq 1$,

$$b_{k,0} = \frac{a_{k,0} - Q_{k-1}(\alpha_k)}{\Pi_{k-1}(\alpha_k)},$$

$$b_{k,l} = \frac{a_{k,l} - \frac{Q_{k-1}^{(l)}(\alpha_k)}{l!} - \sum_{n=0}^{l-1} \frac{1}{(l-n)!} \Pi_{k-1}^{(l-k)}(\alpha_j) b_{k,n}}{\Pi_{k-1}(\alpha_k)} \quad \text{for } 1 \leq l < m_k$$

where we have denoted by

$$\Pi_k(\xi) = \prod_{n=0}^k (\xi - \alpha_n)^{m_n}.$$

The following lemma describes the image of the map ρ and is crucial for the rest of the proof. It is an easy consequence of [12, Theorem 1.11].

Lemma 4.2. *A doubly indexed sequence $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ belongs to $\rho(\mathcal{G}_\theta(\mathbb{C}))$ if and only if*

$$\sup_{k \in \mathbb{N}} \sup_{0 \leq l < m_k} |b_{k,l}| (|\alpha_k| + 1)^{m_0 + \dots + m_{k-1} + l} e^{-\theta(m|\alpha_k|)} < +\infty,$$

for a certain $m > 0$, where $b = \{b_{k,l}\}_{k, 0 \leq l < m_k} = \Psi^{-1}(a)$.

In order to give a topological structure, let us denote by $\mathcal{B}_{\theta,m}(V)$ the Banach space of all doubly indexed sequences of complex numbers $b = \{b_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that

$$\|b\|_{\theta,m} = \sup_{k \in \mathbb{N}} \sup_{0 \leq l < m_k} |b_{k,l}| (|\alpha_k| + 1)^{m_0 + \dots + m_{k-1} + l} e^{-\theta(m|\alpha_k|)} < +\infty.$$

Let us consider the space $\mathcal{A}_{\theta,m}(V) = \Psi^{-1}(\mathcal{B}_{\theta,m}(V))$, that is, the space of all doubly indexed sequences of complex numbers $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that

$$\|\Psi(a)\|_{\theta,m} < +\infty.$$

It is easy to see that $\mathcal{A}_{\theta,m}(V)$ endowed with the norm $\|a\|_{\theta,m} = \|\Psi(a)\|_{\theta,m}$ is a Banach space and that Ψ is an isometry from $\mathcal{A}_{\theta,m}(V)$ into $\mathcal{B}_{\theta,m}(V)$. Now, we define the spaces

$$\mathcal{A}_\theta(V) = \cup_{p \in \mathbb{N}^*} \mathcal{A}_{\theta,p}(V) \text{ and } \mathcal{B}_\theta(V) = \cup_{p \in \mathbb{N}^*} \mathcal{B}_{\theta,p}(V)$$

endowed with the topology of inductive limit of Banach spaces.

We define the linear map

$$\alpha = \Psi \circ \rho \circ \mathcal{L} : \mathcal{F}'_\theta(\mathbb{C}) \rightarrow \mathcal{B}_\theta(V).$$

Proposition 4.3. *The map α is continuous and surjective.*

Proof. By Proposition 2.4, we know that $\mathcal{L} : \mathcal{F}'_\theta(\mathbb{C}) \rightarrow \mathcal{G}_\theta(\mathbb{C})$ is a topological isomorphism. By Lemma 4.2, the operator

$$\rho : \mathcal{G}_\theta(\mathbb{C}) \rightarrow \mathcal{A}_\theta(V)$$

is surjective. It is also continuous by [12, Proposition 1.8]. Finally, by construction, it is clear that

$$\Psi : \mathcal{A}_\theta(V) \rightarrow \mathcal{B}_\theta(V)$$

is a topological isomorphism. ■

Recall that $\mathcal{F}_\theta(\mathbb{C})$ is a Fréchet-Schwartz space, therefore it is reflexive. Then, the transpose α^t of α is defined from the strong dual of $\mathcal{B}_\theta(V)$, denoted by $\mathcal{B}'_\theta(V)$, into $\mathcal{F}_\theta(\mathbb{C})$.

Next, we need to characterize the dual space $\mathcal{B}'_\theta(V)$ as a space of doubly indexed sequences :

Lemma 4.4. *The space $\mathcal{B}'_\theta(V)$ is topologically isomorphic through the canonical bilinear form*

$$\langle c, b \rangle = \sum_{k=0}^{+\infty} \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}$$

to the space $\mathcal{C}_\theta(V) = \cap_{p \in \mathbb{N}^} \mathcal{C}_{\theta,p}(V)$ endowed with the projective limit topology, where, for all p , $\mathcal{C}_{\theta,p}(V)$ is the Banach space of the sequences $c = \{c_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that*

$$(12) \quad \|c\|'_{\theta,p} := \sum_{k \in \mathbb{N}} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)} \right) < +\infty.$$

Moreover, $\mathcal{C}_\theta(V)$ is a Fréchet-Schwartz space.

Proof. Let us show that $\beta : \mathcal{C}_\theta(V) \rightarrow \mathcal{B}'_\theta(V)$ defined by

$$\langle \beta(c), b \rangle = \langle c, b \rangle = \sum_{k=0}^{+\infty} \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}$$

is a topological isomorphism.

Let $c = \{c_{k,l}\}_{k, 0 \leq l < m_k}$ be an element of $\mathcal{C}_\theta(V)$ and $b = \{b_{k,l}\}_{k, 0 \leq l < m_k} \in \mathcal{B}_{\theta,p}(V)$, for a certain p . For any $k \geq 0$, we have, by definition of $\|b\|_{\theta,p}$,

$$\sum_{l=0}^{m_k-1} |b_{k,l} c_{k,l}| \leq e^{\theta(p|\alpha_k|)} \|b\|_{\theta,p} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)}.$$

Using the estimate (12), we see that the sum converges (absolutely) and that

$$|\langle c, b \rangle| \leq \|c\|'_{\theta,p} \|b\|_{\theta,p}.$$

This shows the continuity of β . Let $B^{k,l}$ be the doubly indexed sequence of \mathbb{C} defined by (using the Kronecker symbols) :

$$(13) \quad B^{k,l} = \{\delta_{kj}\delta_{ln}\}_{j,0 \leq n < m_j}.$$

We easily see that $B^{k,l} \in \mathcal{B}_{\theta,p}(V)$. For all k and $0 \leq l < m_k$, we have $c_{k,l} = \langle \beta(c), B^{k,l} \rangle$. It is then clear that β is injective.

Conversely, to an element $\nu \in \mathcal{B}'_{\theta}(V)$, consider the doubly indexed sequence $c = \{c_{k,l}\}_{k,0 \leq l < m_k}$ defined by

$$c_{k,l} = \langle \nu, B^{k,l} \rangle$$

To verify that $c \in \mathcal{C}_{\theta}(V)$, let $p \in \mathbb{N}^*$ be fixed and define $\tilde{b} = \{\tilde{b}_{k,l}\}_{k,0 \leq l < m_k}$ by

$$\tilde{b}_{k,l} = e^{\theta(p|\alpha_k|)} \frac{\bar{c}_{k,l}}{|c_{k,l}|} (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)} \text{ if } c_{k,l} \neq 0, \quad \tilde{b}_{k,l} = 0 \text{ otherwise.}$$

It is clear that $\tilde{b} \in \mathcal{B}_{\theta,p}(V)$ and that $\|\tilde{b}\|_{\theta,p} \leq 1$. Therefore, all the finite sequences $\tilde{b}^K = \sum_{k=0}^K \sum_{l=0}^{m_k-1} \tilde{b}_{k,l} B^{k,l}$ satisfy

$$\|\tilde{b}^K\|_{\theta,p} \leq 1.$$

Denoting by $\|\nu\|'_{\theta,p}$ the norm in $\mathcal{B}'_{\theta,p}(V)$, we have, for all K ,

$$|\langle \nu, \tilde{b}^K \rangle| \leq \|\nu\|'_{\theta,p} \|\tilde{b}^K\|_{\theta,p} \leq \|\nu\|'_{\theta,p}.$$

On the other hand,

$$\langle \nu, \tilde{b}^K \rangle = \sum_{k=0}^K \sum_{l=0}^{m_k-1} \tilde{b}_{k,l} \langle \nu, B^{k,l} \rangle = \sum_{k=0}^K e^{\theta(p|\alpha_k|)} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)},$$

by definition of $\tilde{b}_{k,l}$. Letting K tend to infinity, we obtain that $c \in \mathcal{C}_{\theta,p}(V)$ and that

$$(14) \quad \|c\|_{\theta,p} \leq \|\nu\|'_{\theta,p}.$$

Consider now an element $b = \{b_{k,l}\}_{k,0 \leq l < m_k}$ of $\mathcal{B}_{\theta,p}(V)$ and put

$$b^K = \sum_{k=0}^K \sum_{l=0}^{m_k-1} b_{k,l} B^{k,l}.$$

Let q be an integer strictly larger than p . Note that by convexity of θ , for all k the following inequality holds

$$(15) \quad -\theta(q|\alpha_k|) + \theta(p|\alpha_k|) \leq -(1 - p/q)\theta(q|\alpha_k|).$$

Using this inequality, we find

$$\|b - b^K\|_{\theta,q} \leq \|b\|_{\theta,p} \sup_{k > K} e^{-\theta(q|\alpha_k|) + \theta(p|\alpha_k|)} \leq \|b\|_{\theta,p} e^{-(1-p/q)\theta(q|\alpha_K|)}.$$

We readily deduce that b^K converges to b when K tends towards infinity and that

$$\langle \nu, b \rangle = \sum_{k=0}^{+\infty} \sum_{l=0}^{m_k-1} b_{k,l} \langle \nu, B^{k,l} \rangle = \langle c, b \rangle.$$

We conclude that $\beta(c) = \nu$ and that β is surjective. The continuity of β^{-1} is a direct consequence of the inequality (14).

In order to prove that $\mathcal{C}_\theta(V)$ is a Fréchet-Schwartz space, in view of [2, Proposition 1.4.8.], it is sufficient to see that, for any $p \in \mathbb{N}^*$, the canonical injection

$$i_p : \mathcal{C}_{\theta,p+1}(V) \rightarrow \mathcal{C}_{\theta,p}(V)$$

is compact. Let $\{c^n\}_n$ be a sequence of elements in $\mathcal{C}_{\theta,p+1}(V)$ such that, for all n , $\|c^n\|_{\theta,p+1} \leq 1$. It suffices to show that one can extract a subsequence of $\{c^n\}_n$ converging in $\mathcal{C}_{\theta,p}(V)$.

It is easy to see that, for all $k \in \mathbb{N}$ and $0 \leq l < m_k$ the sequence $\{c_{k,l}^n\}_n$ is bounded. Thus, up to taking a subsequence, we may assume that $c_{k,l}^n$ converges to a certain $c_{k,l} \in \mathbb{C}$. Putting $c = \{c_{k,l}\}_{k,0 \leq l < m_k}$, we readily see that $c \in \mathcal{C}_{\theta,p+1}(V)$ and $\|c\|_{\theta,p+1} \leq 1$.

Let us verify that $\|c^n - c\|_{\theta,p}$ tends to 0 when n tends to infinity. We assume that $|\alpha_k| \rightarrow \infty$, otherwise, the result is trivial. Then, again using inequality (15) we find that $e^{\theta(p|\alpha_k|) - \theta((p+1)|\alpha_k|)}$ tends to 0 when k tends towards infinity. Let $\varepsilon > 0$. For a certain $K \in \mathbb{N}$ and for all $k \geq K$, $e^{\theta(p|\alpha_k|) - \theta((p+1)|\alpha_k|)} < \frac{\varepsilon}{4}$. Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned} & \sum_{k \geq K} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}^n - c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)} \right) \\ & \leq \frac{\varepsilon}{4} \sum_{k \geq K} e^{\theta((p+1)|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}^n - c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)} \right) \\ & \leq \frac{\varepsilon}{4} \|c^n - c\|_{\theta,p+1} \leq \frac{\varepsilon}{4} (\|c^n\|_{\theta,p+1} + \|c\|_{\theta,p+1}) \leq \frac{\varepsilon}{2}. \end{aligned}$$

Moreover, for a certain $N \in \mathbb{N}$ and for all $n \geq N$, we have

$$\sum_{k=0}^{K-1} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}^n - c_{k,l}| (|\alpha_k| + 1)^{-(m_0 + \dots + m_{k-1} + l)} \right) \leq \frac{\varepsilon}{2}.$$

Finally, for $n \geq N$, $\|c^n - c\|_{\theta,p} < \varepsilon$. ■

From now on, we will identify $\mathcal{B}'_\theta(V)$ with the space $\mathcal{C}_\theta(V)$. The next step is to prove the following lemma :

Lemma 4.5. (i) α^t is a topological isomorphism onto its image and $\text{Im } \alpha^t = (\text{Ker } \alpha)^\circ$, the orthogonal space of $\text{Ker } \alpha$.

(ii) $\text{Ker } \alpha = \{T \star U, U \in \mathcal{F}'_\theta(\mathbb{C})\}$.

(iii) $(\text{Ker } \alpha)^\circ = \text{Ker } T \star = \{f \in \mathcal{F}_\theta(\mathbb{C}) \mid T \star f = 0\}$.

Proof. (i) From Proposition 4.3, α is a surjective continuous linear map. Therefore, α^t is a topological isomorphism onto its image and $\text{Im } \alpha^t = (\text{Ker } \alpha)^\circ$ (see [2, Proposition 1.4.12]).

(ii) Recalling Remark 4.1, we have

$$\text{Ker } \alpha = \text{Ker } (\rho \circ \mathcal{L}) = \mathcal{L}^{-1}(\text{Ker } \rho) = \{T \star \mathcal{L}^{-1}(g), g \in \mathcal{G}_\theta(\mathbb{C})\} = \{T \star U, U \in \mathcal{F}'_\theta(\mathbb{C})\}.$$

(iii) Let f be an element of $(\text{Ker } \alpha)^\circ$. For all $z \in \mathbb{C}$,

$$(T \star f)(z) = \langle T, \tau_z f \rangle = \langle T, \delta_z \star f \rangle = \langle T \star \delta_z, f \rangle = 0,$$

using the fact that $T \star \delta_z \in \text{Ker } \alpha$.

Conversely, let $f \in \mathcal{F}_\theta(\mathbb{C})$ be such that $T \star f = 0$ and let $U \in \mathcal{F}'_\theta(\mathbb{C})$. We have

$$\langle T \star U, f \rangle = \langle U, T \star f \rangle = 0.$$

This shows that $f \in (\text{Ker } \alpha)^\circ$ and concludes the proof of the lemma. ■

Let us proceed with the proof of Theorem 3.5.

(i) Let $f \in \mathcal{F}_\theta(\mathbb{C})$ be a T -mean-periodic function, that is, $f \in \text{Ker } T \star$. From Lemmas 4.5 and 4.4, there is a unique sequence $c \in \mathcal{C}_\theta(V)$ such that $f = \alpha^t(c)$.

For $z \in \mathbb{C}$, denoting by δ_z the Dirac measure at z , we have

$$f(z) = \langle \delta_z, f \rangle = \langle \delta_z, \alpha^t(c) \rangle = \langle c, \alpha(\delta_z) \rangle = \langle c, \Psi(\rho(g_z)) \rangle$$

where we have denoted by $g_z = \mathcal{L}(\delta_z)$, that is, the function in $\mathcal{G}_\theta(\mathbb{C})$ defined by $g_z(\xi) = e^{z\xi}$.

Let us compute $\Psi(\rho(g_z)) = b(z) = \{b_{k,l}(z)\}_{k,0 \leq l < m_k}$, which is an element of $\mathcal{B}_\theta(V)$. By well know formulas about Newton polynomials (See, for example [2, Definition 6.2.8]), we have, for $k \in \mathbb{N}$, and denoting by

$$\partial_j^m = \frac{1}{m!} \frac{\partial^m}{\partial \alpha_j^m},$$

for $0 \leq l < m_k$,

$$b_{k,l}(z) = \partial_0^{m_0-1} \dots \partial_{k-1}^{m_{k-1}-1} \partial_k^l \left(\sum_{j=0}^k e^{z\alpha_j} \prod_{0 \leq n \leq k, n \neq j} (\alpha_j - \alpha_n)^{-1} \right) = \sum_{j=0}^k e^{z\alpha_j} P_{k,j,l}(z),$$

where we have denoted by, for $j < k$,

$$(16) \quad P_{k,j,l}(z) = \sum_{i=0}^{m_j-1} \frac{z^i}{i!} \partial_i^{m_j-1-i} \left(\prod_{0 \leq n \leq k-1, n \neq j} (\alpha_j - \alpha_n)^{-m_n} (\alpha_j - \alpha_k)^{-(l+1)} \right)$$

and

$$(17) \quad P_{k,k,l}(z) = \sum_{i=0}^l \frac{z^i}{i!} \partial_k^{l-i} \left(\prod_{0 \leq n \leq k-1} (\alpha_k - \alpha_n)^{-m_n} \right).$$

Thus,

$$f(z) = \sum_{k \geq 0} \left(\sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right) = \sum_{k \geq 0} \left(\sum_{l=0}^{m_k-1} c_{k,l} \sum_{j=0}^k e^{z\alpha_j} P_{k,j,l}(z) \right)$$

and the equality (7) is established. Let us now verify the convergence in $\mathcal{F}_\theta(\mathbb{C})$ of the series.

Case where $\theta(x) = x$. Here, $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{H}(\mathbb{C})$. We have to verify that the serie converges uniformly on every compact of \mathbb{C} . Let $p \in \mathbb{N}^*$ and $z \in \mathbb{C}$, $|z| \leq p$.

We have, for all $\xi \in \mathbb{C}$, $|g_z(\xi)| = |e^{z\xi}| \leq e^{p|\xi|}$, that is,

$$\|g_z\|_{\theta,p} \leq 1.$$

Thus, by continuity of $\Psi \circ \rho$, there exists $p' \in \mathbb{N}^*$ and $C_p > 0$ such that

$$\|b(z)\|_{\theta,p'} \leq C_p \|g_z\|_{\theta,p} \leq C_p.$$

For all $k \geq 0$, we have

$$\sum_{l=0}^{m_k-1} |c_{k,l} b_{k,l}(z)| \leq \|b(z)\|_{\theta,p'} e^{\theta(p'|\alpha_k|)} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_0+\dots+m_{k-1}+l)}$$

We obtain

$$\sup_{|z| \leq p} \sum_{l=0}^{m_k-1} |c_{k,l} b_{k,l}(z)| \leq C_p e^{\theta(p'|\alpha_k|)} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_0+\dots+m_{k-1}+l)}$$

Recalling that $c \in \mathcal{C}'_\theta(V)$, the right term is the general term of a convergent serie, thus, the right-hand side of (7) is convergent in $\mathcal{F}_\theta(\mathbb{C})$. Moreover,

$$\sup_{|z| \leq p} \sum_{k \geq 0} \sum_{l=0}^{m_k-1} |c_{k,l} b_{k,l}(z)| \leq C_p \|c\|'_{\theta,p'}.$$

Case where θ is a Young function. For any $p \in \mathbb{N}^*$, observe that

$$\|g_z\|_{\theta,p} \leq e^{\theta^*(\frac{1}{p}|z|)}.$$

Thus, by continuity of $\Psi \circ \rho$, there exists $p' \in \mathbb{N}^*$ and $C_p > 0$ such that

$$\|b(z)\|_{\theta,p'} \leq C_p \|g_z\|_p \leq C_p e^{\theta^*(\frac{1}{p}|z|)}.$$

For all $k \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

$$\sum_{l=0}^{m_k-1} |c_{k,l} b_{k,l}(z)| \leq \|b(z)\|_{\theta,p'} e^{-\theta(p'|\alpha_k|)} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{m_0+\dots+m_{k-1}+l}$$

We obtain

$$\sup_{z \in \mathbb{C}} \sum_{l=0}^{m_k-1} |c_{k,l} b_{k,l}(z)| e^{-\theta^*(\frac{1}{p}|z|)} \leq C_p e^{-\theta(p'|\alpha_k|)} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{m_0+\dots+m_{k-1}+l}.$$

As in the previous case, we deduce that the right-hand side of (7) is absolutely convergent in $\mathcal{F}_\theta(\mathbb{C})$. Moreover,

$$\sup_{z \in \mathbb{C}} \sum_{k \geq 0} \sum_{l=0}^{m_k-1} |c_{k,l} b_{k,l}(z)| \leq C_p \|c\|'_{\theta,p'}.$$

In order to find an explicit formula for the coefficients $c_{n,l}$, consider the elements $B^{k,l}$ of $\mathcal{B}_\theta(\mathbb{C})$ defined by (13) and observe that, by the definition of the Newton polynomials (see (11)) with respect to the coefficients of $B^{k,l}$, for all $q \geq k$, we have

$$Q_q(\xi) = (\xi - \alpha_k)^l \prod_{l=0}^{k-1} (\xi - \alpha_l)^{m_l}$$

and for $q < k$, $Q_q = 0$. We readily deduce that $\alpha(S_{k,l}) = \Psi \circ \rho \circ \mathcal{L}(S_{k,l}) = B^{k,l}$.

Now, for all $k \in \mathbb{N}$ and $0 \leq l < m_k$,

$$\langle S_{k,l}, f \rangle = \langle S_{k,l}, \alpha^t(c) \rangle = \langle \alpha(S_{k,l}), c \rangle = \langle B^{k,l}, c \rangle = c_{k,l}.$$

(ii) The converse part is easily deduced from the estimates in the proof of (i) and Lemma 3.4.

5. CASE WHERE V IS AN INTERPOLATING VARIETY.

Definition 5.1. We say that V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$ if, for any doubly indexed sequence $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that, for a certain $m > 0$,

$$\sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| e^{-\theta(m|\alpha_k|)} < +\infty,$$

there exists a function $g \in \mathcal{G}_\theta(\mathbb{C})$ such that, for all k and all $0 \leq l < m_k - 1$,

$$\frac{g^l(\alpha_k)}{l!} = a_{k,l}.$$

We assume from now on that V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$. Then we have the following result :

Theorem 5.2. (i) Any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(18) \quad f(z) = \sum_{k \geq 0} e^{z\alpha_k} \sum_{l=0}^{m_k-1} d_{k,l} \frac{z^l}{l!},$$

where the coefficients $a_{k,l}$ verify the following estimate :

$$(19) \quad \sum_{k \geq 0} e^{\theta(m|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |d_{k,l}| \right) < +\infty$$

for every $m > 0$. Moreover, for all $k \in \mathbb{N}$ and $0 \leq l < m_k$, we have the equality

$$d_{k,l} = \langle T_{k,l}, f \rangle$$

where $T_{k,l} \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(T_{k,l})(\xi) = \frac{m_k!}{\Phi^{(m_k)}(\alpha_k)} \frac{\Phi(\xi)}{(\xi - \alpha_k)^{m_k-l}},$$

(ii) Conversely, any such series whose coefficients $d_{k,l}$ satisfy these estimate (19) converges in $\mathcal{F}_\theta(\mathbb{C})$ to a function f solving the equation (6).

Note that $\mathcal{L}(T_{k,l}) \in \mathcal{G}_\theta(\mathbb{C})$ by Proposition 3.2.

Remark 5.3. In the case where $\theta(x) = x$, this Theorem 5.2 is also a consequence of [2, Theorem 6.2.6.].

We will denote by $\mathcal{A}_{\theta,m}(V)$ the space of all doubly indexed sequences of complex numbers $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that

$$(20) \quad \|a\|_{\theta,m} := \sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| e^{-\theta(m|\alpha_k|)} < +\infty$$

and

$$\mathcal{A}_{\theta}(V) = \cup_{p \in \mathbb{N}^*} \mathcal{A}_{\theta,p}(V)$$

endowed with the strict inductive limit of Banach spaces.

We define the linear map

$$\alpha = \rho \circ \mathcal{L} : \mathcal{F}_{\theta}'(\mathbb{C}) \rightarrow \mathcal{A}_{\theta}(V).$$

Proposition 5.4. *The map α is continuous and surjective.*

Proof. It is sufficient to show that the map $\rho : \mathcal{G}_{\theta}(\mathbb{C}) \rightarrow \mathcal{A}_{\theta}(V)$ is surjective and continuous. The surjectivity follows from the fact that V is an interpolating variety.

In order to show the continuity, let $g \in \mathcal{G}_{\theta,p}(\mathbb{C})$ and let $z \in \mathbb{C}$. By the Cauchy estimates applied to the disc of center z and radius 2, for all $l \in \mathbb{N}$,

$$\left| \frac{g^l(z)}{l!} \right| \leq \frac{1}{2^l} \sup_{|\xi-z| \leq 2} |g(\xi)|.$$

For $|\xi - z| \leq 2$, we have

$$|g(\xi)| \leq \|g\|_{\theta,p} e^{\theta(p|\xi|)} \leq \|g\|_{\theta,p} e^{\theta(2p+p|z|)} \leq \|g\|_{\theta,p} e^{1/2\theta(4p)} e^{1/2\theta(2p|z|)}$$

by convexity of θ . Thus,

$$\sum_{l=0}^{\infty} \left| \frac{g^l(z)}{l!} \right| \leq 2 \|g\|_{\theta,p} e^{1/2\theta(2p)} e^{\theta(2p|z|)}.$$

In particular, we deduce that $\rho(g) \in \mathcal{A}_{\theta,2p}(V)$ and that

$$\|\rho(g)\|_{\theta,2p} \leq 2 \|g\|_{\theta,p} e^{1/2\theta(2p)}.$$

The continuity of ρ follows from the last inequality. ■

Remark 5.5. By a standard result about interpolating varieties (see [2, chapter 2]), the multiplicities verify, for certain constants $A, m > 0$,

$$m_k \leq A e^{\theta(m|\alpha_k|)}, \quad \forall k \in \mathbb{N}.$$

Consequently, we may replace the norm given by (20) by the following

$$\|a\|_{\theta,m} := \sup_{k \in \mathbb{N}} \sup_{0 \leq l < m_k-1} |a_{k,l}| e^{-\theta(m|\alpha_k|)}$$

in the definition of $\mathcal{A}_{\theta}(V)$.

Lemma 5.6. *The space $\mathcal{A}'_\theta(V)$ is topologically isomorphic to the space $\mathcal{D}_\theta(V) = \cap_{p \in \mathbb{N}^*} \mathcal{D}_{\theta,p}(V)$ endowed with the projective limit topology, where, for all p , $\mathcal{D}_{\theta,p}(V)$ is the Banach space of the sequences $d = \{d_{k,l}\}_{k,0 \leq l < m_k}$ such that*

$$(21) \quad \|d\|'_{\theta,p} := \sum_{k \geq 0} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |d_{k,l}| \right) < +\infty.$$

Moreover, $\mathcal{D}_\theta(V)$ is a Fréchet-Schwartz space.

In view of the preceding remark, the proof is similar to the one of Lemma 4.4. We are now ready to prove Theorem 5.2. From Lemma 4.5 (which is still valid with the new definition of ρ) and Lemma 5.6, any T -mean-periodic function f is the image by α^t of a unique $d \in \mathcal{A}'_\theta(V)$. We have, for all $z \in \mathbb{C}$,

$$f(z) = \langle \delta_z, f \rangle = \langle \delta_z, \alpha^t(d) \rangle = \langle d, \alpha(\delta_z) \rangle = \langle d, \rho(g_z) \rangle = \sum_{k \geq 0} e^{z\alpha_k} \sum_{l=0}^{m_k-1} \frac{z^l}{l!} d_{k,l}.$$

To compute the coefficients $d_{k,l}$:

$$\langle T_{k,l}, f \rangle = \langle T_{k,l}, \alpha^t(d) \rangle = \langle d, \alpha(T_{k,l}) \rangle = d_{k,l}.$$

The last equality follows from the observation that

$$\alpha(T_{k,l}) = B^{k,l}.$$

The rest of the proof is similar to the one of Theorem 3.5.

Let us recall some results about interpolating varieties that enables one to determine whether V is interpolating or not. We first give a known analytic characterization (see [6] or [2]). The spaces of entire functions considered are slightly different, but it is clear how to adapt these results to our spaces.

Theorem 5.7. *V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$ if and only if, there are constants $\varepsilon > 0$ and $m > 0$ such that, for all k ,*

$$\left| \frac{\Phi^{m_k}(z)}{m_k!} \right| \geq \varepsilon e^{-\theta(m|\alpha_k|)}.$$

We also give a known geometric characterization (see [3, Corollary 4.8] or [11, Theorem 1.8]) in terms of the distribution of the points $\{(\alpha_k, m_k)\}_k$.

Define the counting function and the integrated counting function :

Definition 5.8. For $z \in \mathbb{C}$ and $r > 0$,

$$n(z, r) = \sum_{|z - \alpha_k| \leq r} m_k,$$

$$N(z, r) = \int_0^r \frac{n(z, t) - n(z, 0)}{t} dt + n(z, 0) \ln r = \sum_{0 < |z - \alpha_k| \leq r} m_k \ln \frac{r}{|z - \alpha_k|} + n(z, 0) \ln r.$$

Theorem 5.9. *V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$ if and only if conditions*

$$(22) \quad \exists A > 0, \exists m > 0 \quad \forall R > 0, \quad N(0, R) \leq A + \theta(mR)$$

and

$$(23) \quad \exists A > 0, \exists m > 0 \quad \forall k \in \mathbb{N}, \quad N(\alpha_k, |\alpha_k|) \leq A + \theta(m|\alpha_k|)$$

hold.

Actually, in this paper, since $V = \Phi^{-1}(0)$ and $\Phi \in \mathcal{G}_\theta(\mathbb{C})$, condition (22) is necessarily verified (see, for example, [12, Theorem 1.13]). Thus, V is an interpolating variety if and only if condition (23) holds.

Remark 5.10. We can obtain Theorem 5.2 as a corollary of Theorem 3.5, using the density condition (23). This second proof is rather technical, we will skip it here. Let us just give the correspondence between the coefficients $c_{k,l}$ and $d_{k,l}$:

$$(24) \quad \begin{aligned} d_{k,l} = & \sum_{i=l}^{m_k-1} c_{k,i} \partial_k^{i-l} \left(\prod_{0 \leq n \leq k-1} (\alpha_k - \alpha_n)^{-m_n} \right) \\ & + \sum_{j=k+1}^{\infty} \sum_{i=0}^{m_j-1} c_{j,i} \partial_k^{m_k-1-l} \left(\prod_{0 \leq n \leq j-1, n \neq k} (\alpha_k - \alpha_n)^{-m_n} (\alpha_k - \alpha_j)^{-(i+1)} \right), \end{aligned}$$

the convergence of the second sum being a consequence of conditions (23) and (22).

In the case where all $m_k = 1$, we have

$$d_k = \sum_{j=k}^{\infty} c_j \prod_{0 \leq n \leq j, n \neq k} (\alpha_k - \alpha_n)^{-1}.$$

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